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Probabilistic Time and the Quantum Gravity Interpretation

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We analyze the perturbed minisuperspace models of quantum gravity through the analogy with the time-independent Schrödinger equation. We show that a time variable defined in a previous work, the "*probabilistic time*," is the variable which yields the backreaction Einstein equations.

1. INTRODUCTION

In quantum gravity, the state of the universe is described by a unique wave function ψ which satisfies the Wheeler–De Witt equation (WDWE). Since the proposal of Hartle and Hawking (1983) for giving boundary conditions which univocally specify the wave function, there has been renewed interest in this type of approach to quantize gravity, particularly in its application to quantum cosmology (Hartle, 1986). Nevertheless, one of the fundamental problems to clarify the meaning of ψ has not been completely solved. The point of boundary conditions is now also under discussion (Vilenkin, 1988); we will not address this here. We will suppose that some boundary conditions are given in order to fix a particular solution of the WDWE.

Recently Castagnino (1989) showed that working with a fixed topology, it is possible to introduce a normalization of the wave function (which allows a probabilistic interpretation) and a variable which plays the role of time: the *probabilistic time* θ. This variable is exactly defined in terms of the wave function, it is not a classical or semiclassical object. Thus, we hope it will be relevant in the interpretation of the different cosmological

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models, since the definition of a satisfactory time variable is important to analyze the beginning of the universe.

Castagnino (1989) showed that, to the lowest order in the Planck length σ, θ becomes the classical time which enters into the Schrödinger equation for the matter degrees of freedom. Here we will continue with this approach, showing that, to next order, the probabilistic time yields the semiclassical (backreaction) Einstein equations of the quantum field theory in curved space. Before doing this, we will state our interpretation of the wave function using an analogy between the WDWE and the time-independent Schrödinger equation (TISE).

2. WDWE AND ORDINARY QUANTUM MECHANICS

To begin with, let us consider a model with a single gravitational degree of freedom treated exactly and an infinite number of degrees of freedom representing metric and matter fluctuations which are treated perturbatively (Wada, 1986). The former degree of freedom is typically the scale factor in Robertson-Walker universes. The WDWE is, in this case,

$$\{-\tfrac{1}{2}\sigma^2\nabla^2+\sigma^{-2}U(a)+h(a,\phi_n)\}\Psi(a,\phi_n)=0 \tag{1}$$

where a is the scale factor, ϕ_n, $n=1,2,\ldots$, denote the fluctuations,

$$\nabla^2=(-G)^{-1/2}\frac{d}{da}\left[G^{aa}(-G)^{1/2}\frac{d}{da}\right]$$

and G_{ij} is the minisuperspace metric (in this particular example it has only one component $G_{aa}=-G=-a$); $h(a,\phi_n)$ is the Hamiltonian of the fluctuations. It is important to note that $\psi(a,\phi_n)$ does not depend on an external time parameter: in quantum gravity, time is an observable and the wave function contains all the observables of the universe, including the positions of the hands of all possible clocks.

Equation (1) (or its generalization to more than one gravitational degree of freedom) admits at least two different interpretations: on one hand, since the minisuperspace metric is hyperbolic, it can be viewed as a *mass-dependent Klein-Gordon equation.* On the other hand, since all the variables that appear in the wave function and in the differential operator are observables, it looks like a *TISE* in ordinary quantum mechanics. We will adopt here the latter point of view (Kandrup, 1988). In this context, the natural normalization and interpretation of the wave function are

$$\langle\psi,\psi\rangle=\int|\psi|^2(-G)^{1/2}\,d\phi_n\,da \tag{2}$$

$$dP(a,\phi_n)=|\psi|^2(-G)^{1/2}\,d\phi_n\,da \tag{3}$$

where dP is the probability to find the universe in the volume element

$(-G)^{1/2}\, d\phi_n\, da$. Equation (3) is valid only if the wave function is normalized, i.e., if $\langle\psi, \psi\rangle = 1$, otherwise both quantities are proportional. The conditional probability $dP(\phi_n/a)$ to observe the values ϕ_n given the radius of the universe is

$$dP(\phi_n/a) = |\psi|^2(-G)^{1/2}\, d\phi_n \Big/ \int d\phi_n\, |\psi|^2(-G)^{1/2} \tag{4}$$

The main objection to this point of view is that, since in the normalization (2) one is integrating over all variables (including the one which will play the role of "time"), it is likely that one cannot normalize the wave function (Vilenkin, 1988); note that in ordinary quantum mechanics $\int dq\, dt\, |\psi(q, t)|^2 = \infty$. In addition, it is not clear how to extract a "time variable" and a conserved probability for the fluctuations. We will address now these issues.

From equation (3) it is easy to obtain the probability $dP(\theta, \phi_n)$, where $\theta = \theta(a)$ is an arbitrary *increasing* function of a (this assumption will imply a "cosmological arrow of time"). It reads

$$dP(\theta, \phi_n) \sim d\phi_n\, d\theta\, |\psi|^2(-G)^{1/2}\, da/d\theta \tag{5}$$

The function $\theta(a)$ can be fixed by imposing that $dP(\theta, \phi_n)$ be independent of θ when the fluctuations are integrated, i.e.,

$$\int d\phi_n\, |\psi|^2(-G)^{1/2}\, da/d\theta = (\text{const})^{-1} \tag{6a}$$

or

$$d\theta/da = \text{const} \cdot \int d\phi_n\, |\psi|^2(-G)^{1/2} \tag{6b}$$

This is just the definition of "probabilistic time" given in Castagnino (1989) using a different argument.

The function $\chi(\theta, \phi_n)$, defined up to a phase through

$$\chi(\theta, \phi_n) = (\text{const})^{1/2}\psi(-G)^{1/4}(da/d\theta)^{1/2} \tag{7a}$$

is automatically normalized (for all θ) in the following sense:

$$(\chi, \chi) = \int |\chi|^2\, d\phi_n = 1 \tag{7b}$$

Note that we have two different normalizations denoted by $\langle , \rangle$ and $(,)$. In the first one the integration is performed over all variables, while in the second one θ is not integrated, giving a θ-independent normalization for χ.

In terms of the variable θ, equation (2) becomes

$$\langle \psi, \psi \rangle \equiv \int |\chi|^2 \, d\phi_n \, d\theta = \int d\theta = \theta_{\max} - \theta_{\min} \tag{8}$$

and will be infinite or not, depending on the model considered. For example, if the model has a classical limit, then $\langle \psi, \psi \rangle$ will be infinite because $\theta_{\max} \sim t_{\max} = \infty$.

Now we see what is going on. The introduction of the variable θ allowed us to extract from ψ a probability density ($|\chi|^2$) with conserved normalization and also to isolate the possible source of nonnormalizability of the complete wave function [see equation (8)]. The analogy with the TISE is useful to clarify the last point. The TISE admits two types of solutions: normalizable solutions with quantized values of the energy and nonnormalizable ones with continuous spectrum. This situation translates *mutatis mutandi* to the WDWE. There will be normalizable solutions (with quantized cosmological) constant and nonnormalizable ones, depending on the model and boundary conditions considered. Some examples of the first type have been analyzed in Wudka (1987), although they may appear unphysical. The nonnormalizable solutions are more relevant, since they give the correct classical behavior.

It is important to note that the nonnormalizability of ψ is not an obstacle to calculating the conditional probabilities [equation (4)] and to introducing the probabilistic time θ and the wave function χ. Furthermore, one can insist (Castagnino, 1989) and normalize ψ working in a finite "temporal box" of size $T = (\theta_{\max} - \theta_{\min})$ and dividing the wave function by the factor $T^{1/2}$. This is analogous to what is done, for instance, for free particles in ordinary quantum mechanics, where one divides $\psi(\mathbf{x}) = \exp(i\mathbf{p} \cdot \mathbf{x})$ by the square root of the total volume in order to normalize the wave function.

The following point is also worth noting. Our procedure depends on the choice of the variable in terms of which the probabilistic time is defined. In the example we studied above, the natural candidate was the radius of the universe and, in fact, we defined θ in terms of a. This is justified because one expects a to be the first variable which becomes semiclassical. The case in which several gravitational degrees of freedom are treated exactly deserves future study.

Finally, the interpretation of the wave function based on the analogy with the Klein-Gordon equation has been analyzed by Vilenkin (1988). It can be shown that, in the *semiclassical limit*, both approaches give equivalent results (Castagnino, 1989). The main point is that, in order to obtain a probability density with conserved normalization [see equations (7)], the wave function ψ must be multiplied by a factor. In our case, this factor is supplied by the definition of the probabilistic time, while in Vilenkin (1988) the same factor appears due to the definition of the Klein-Gordon-conserved probability current.

3. THE PROBABILISTIC TIME AND THE BACKREACTION EQUATIONS

It is well known that (Hartle, 1986), in the semiclassical limit, the WDWE yields both the classical Einstein equations for the metric of space-time and the Schrödinger equation for the fluctuations. The latter is nothing but the Schrödinger representation of the quantum field theory in curved space. To next order, the Einstein equations are modified and become the so-called "backreaction equations" (BE), in which the mean value of the energy-momentum tensor of the fluctuation enters as a source. The first derivation of the BE has been given in Hartle (1986), where it is shown that the BE follow after *postulating* a correlation between coordinates and momenta. A more rigorous derivation based on the use of the Wigner function to *compute* the correlation was presented in Halliwell (1987). An alternative derivation should be possible in terms of the probabilistic time because, as we defined a time parameter to all orders in σ, the relation between the coordinate a and its "temporal derivative" $da/d\theta$ is fixed from the beginning. We now discuss this point.

Let us solve the WDWE in powers of σ considering a wave function of the form

$$\psi(a, \phi_n) = e^{iK(a)}J(a, \phi_n) \tag{9}$$

where $K = \sigma^{-2}K_0 + K_1 + \cdots$ and $J = J_0 + \sigma^2 J_1 + \cdots$ (note that K_1 and K_2 are arbitrary since they can be included into J_0 and J_1). Setting equation (9) into equation (1), we get, to lowest order,

$$\frac{1}{2a}\left(\frac{dK_0}{da}\right)^2 - U(a) = 0 \tag{10a}$$

i.e., K_0 satisfies the classical Hamilton-Jacobi equation. The next two orders give the following equations:

$$\frac{2J_0}{a}\frac{dK_0}{da}\frac{dK_1}{da} - \frac{2i}{a}\frac{dK_0}{da}\frac{\partial J_0}{\partial a} - i\frac{J_0}{a}\frac{d^2K_0}{da^2}$$

$$+\frac{i}{2a^2}J_0\frac{dK_0}{da} - 2hJ_0 = 0 \tag{10b}$$

$$\frac{1}{a}\left(2J_1\frac{dK_0}{da}\frac{dK_1}{da}+J_0\left(\frac{dK_1}{da}\right)^2+2J_0\frac{dK_0}{da}\frac{dK_2}{da}\right.$$
$$\left.-2i\frac{dK_0}{da}\frac{\partial J_1}{\partial a}-2i\frac{dK_1}{da}\frac{\partial J_0}{\partial a}-iJ_0\frac{d^2K_1}{da^2}-iJ_1\frac{d^2K_0}{da^2}-\frac{\partial^2 J_0}{\partial a^2}\right)$$
$$+\frac{1}{2a^2}\left(iJ_0\frac{dK_1}{da}+iJ_1\frac{dK_0}{da}+\frac{\partial J_0}{\partial a}\right)-2hJ_1=0 \tag{10c}$$

On the other hand, from the definition of probabilistic time we have

$$\frac{d\theta}{da}=a^{1/2}\,e^{-2\,\mathrm{Im}\,K_1}\{(J_0,J_0)+2\sigma^2[\mathrm{Re}(J_0,J_1)-(J_0,J_0)\,\mathrm{Im}\,K_2]+\cdots\} \tag{11}$$

We will use the freedom in K_1 and K_2 in order to make J coincide with the function χ introduced in the previous section. From equations (7) and (11) we see that $|J|=|\chi|$ if $(J_0, J_0)=1$ and $\mathrm{Re}(J_0, J_1)=0$. Imposing these conditions (which fix Im K_1 and Im K_2) we find, after standard manipulations (Hartle, 1986; Castagnino, 1989; Halliwell, 1987)

$$e^{2\,\mathrm{Im}\,K_1}=a^{-1/2}|dK_0/da| \tag{12a}$$

$$-\frac{i}{a}\frac{dK_0}{da}\frac{\partial J_0}{\partial a}=hJ_0 \tag{12b}$$

where we also set Re $K_1=0$ in order to get (12b). Other choices give rise to an additional term in this equation (Hartle, 1986; Castagnino, in press).

From equations (11) and (12a) we find, to lowest order,

$$\frac{da}{d\theta}=\frac{1}{a}\left|\frac{dK_0}{da}\right| \tag{13}$$

Replacing this result into equations (10a) and (12b), we find

$$\frac{a}{2}\left(\frac{da}{d\theta}\right)^2-U(a)=0 \tag{14a}$$

$$i\frac{\partial J_0}{\partial \theta}=hJ_0 \tag{14b}$$

this is the result of Castagnino (1989): to order σ^0, θ becomes the classical time and one obtains the 0–0 component of the classical Einstein equations (14a) and the usual Schrödinger equation for the fluctuations (14b).

To compute θ in the next order, it is necessary to evaluate Im K_2. Multiplying equation (10c) by J_0^*, integrating the variables ϕ_n, taking the imaginary part, and using that

$$(hJ_0, J_1)=(J_0, hJ_1)=\frac{i}{a}\frac{dK_0}{da}\left(\frac{\partial J_0}{\partial a}, J_1\right) \tag{15a}$$

$$\left(J_0, \frac{\partial J_0}{\partial a}\right) = \frac{ia(J_0, hJ_0)}{dK_0/da} \tag{15b}$$

$$\operatorname{Im}\left(J_0, \frac{\partial^2 J_0}{\partial a^2}\right) = \frac{\partial}{\partial a}\left(\frac{a(J_0, hJ_0)}{dK_0/da}\right) \tag{15c}$$

one gets, after some calculations,

$$\frac{d \operatorname{Im} K_2}{da} = \frac{1}{2}\frac{d}{da}\left[\frac{a(J_0, hJ_0)}{(dK_0/da)^2}\right] \tag{16}$$

From equations (16) and (12) we find

$$\frac{da}{d\theta} = \frac{1}{a}\left|\frac{dK_0}{da}\right|\left[1 + \frac{\sigma^2 a(J_0, hJ_0)}{(dK_0/da)^2} + \cdots\right] \tag{17}$$

so, using equation (10a),

$$\tfrac{1}{2}a\left(\frac{da}{d\theta}\right)^2 - U = \sigma^2(J_0, hJ_0) \tag{18}$$

This equation is the 0–0 component of the BE, since the left-hand side is the classical one [see equation (14a)], while the correction in the right-hand side is the mean value of the energy of the fluctuations. This is a *nontrivial property* of the probabilistic time: while in the classical limit (order σ^0) $a(\theta)$ satisfies the classical Einstein equation, including the first correction in σ^2, we proved that $a(\theta)$ is the solution of the 0–0 backreaction equation. We see then that the variable θ has a meaning *beyond the classical limit*, in which it becomes the classical time.

We think that this is a very interesting result, so we plan to continue the investigation using this approach by analyzing the probabilistic time in more realistic models, including, for instance, more gravitational degrees of freedom, and also by computing it up to higher orders in the Planck length. This should be useful in understanding the cosmological models near the initial singularity.

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